This result, it is evident, is equally true if some terms vanish.

In the first part of this demonstration we had the condition that L_n must not pass through the value of Sx^n. But as we have since proved that this cannot be the case under the conditions in 1^0, 2^0, 3^0, we may erase this condition and we have the following

General Result. In the expansion of $f(x)$ by Maclaurin's Theorem, unless $f(x)$ becomes imaginary or infinite as x increases from 0 to x_1 the sum of the series approaches the true value of $f(x)$—

1. If the signs of the terms are the same;
2. If the signs alternate, and the series decreases;
3. If m terms of one sign follow n terms of the other, and the sum of the former is numerically less than that of the latter.
4. If the series terminates.

[To be continued.]

SOME EXAMPLES OF A NEW METHOD OF SOLVING PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

COMPILED BY GEORGE EASTWOOD, SAXONVILLE, MASS.

EXAMPLE 1. It is required to formulate a method for integrating partial differential equations of the second order, with variable coefficients, that shall facilitate their integration from the terms of the first order. It is also required to apply the method to the integration of the equation

$$\frac{\frac{d^2u}{dt^2}}{t^2}-\frac{\frac{d^2u}{dx^2}}{x^2}=\frac{\frac{du}{dt}}{t^3}-\frac{\frac{du}{dx}}{x^3}+P.$$

Solution.—First, as to the required method. Let u be any function of t and x. Then since

$$\frac{du}{dt}=\left(\frac{du}{dt}\right)+\frac{du}{dx}\cdot\frac{dx}{dt},$$

let it be put in the symbolic form

$$D_tu=d_tu+D_tx.d_xu. \tag{α}$$

Let equation (α) be differentiated with the symbol $\varDelta_t$ so that

$$\varDelta_tD_tu=\varDelta_t(d_tu)+\varDelta_t(D_tx).d_xu+D_tx.\varDelta_t(d_xu). \tag{β}$$

Now if in equation (α) we place d_tu and d_xu successively for u, we shall find

$$D_t.d_tu = d_t.d_tu + D_tx.d_xd_tu = d_t^2u + D_tx.d_t.d_xu,$$
$$D_t.d_tx = d_t.d_xu + D_tx.d_xd_xu = d_t.d_xu + D_tx.d_x^2u.$$

We shall also find, D_t and Δ_t being commutative,

$$\Delta_t(d_tu) = d_t\Delta_tu = d_tD_tu = d_t^2u + D_tx.d_td_xu,$$
$$\Delta_t(D_tx).d_xu = D_t.\Delta_tx.d_xu = \Delta_tx(d_td_xu + D_tx.d_x^2u),$$
$$D_tx.\Delta_t(d_xu) = D_tx.d_x.\Delta_tu = \Delta_t.D_tx.d_xu.$$

Making these substitutions in equation (β), we shall have

$$\Delta_tD_tu = d_t^2u + (D_tx + \Delta_tx)d_td_xu + D_tx.\Delta_tx.d_x^2u + \Delta_tD_tx.d_xu. \qquad (\gamma)$$

Equation (γ) is a very important one; it enables us to solve a certain class of equations with variable coefficients and to eliminate, at one step, all terms of the second order by substituting

$$\Delta_tD_tu - \Delta_tD_tx.d_xu \qquad (\delta)$$

for them. If therefore, in any equation of the second order, with variable coefficients, we substitute the expression (δ) we shall find a new feature of integration-power that will furnish us with the required method.

Secondly, as to the application of the method. Multiply the proposed equation by t^2, then

$$\frac{d^2u}{dt^2} - \frac{t^2}{x^2}\cdot\frac{d^2u}{dx^2} - \frac{1}{t}\cdot\frac{du}{dt} + \frac{t^2}{x^3}\cdot\frac{du}{dx} = Pt^2,$$

which expressed symbolically gives

$$d_t^2u - \frac{t^2}{x^2}.d_x^2u - \frac{1}{t}.d_tu + \frac{t^2}{x^3}.d_xu = Pt^2.$$

From equation (γ) we have

$$\Delta_tD_tu - \Delta_tD_tx.d_xu = d_t^2u + (D_tx + \Delta_tx)d_td_xu + D_tx.\Delta_tx.d_x^2u.$$

Comparing this with the proposed example we find that it will agree with the terms of the second order if we assume

$$D_tx = -\Delta_tx = t \div x.$$

Now these assumptions give, by differentiation and integration,

$$x^2 = t^2 + \xi,$$
$$x^2 = -t^2 + \xi',$$
$$\Delta_tD_tx = \Delta_t\times(t\div x)$$
$$= \frac{1}{x} + \frac{t^2}{x^3}.$$

Hence, by substitution,

$$\Delta_tD_tu - \left(\frac{1}{x} + \frac{t^2}{x^3}\right)d_xu - \frac{1}{t}.d_tu + \frac{t^2}{x^3}.d_xu = Pt^2.$$

$$\therefore \qquad \Delta_tD_tu - \frac{1}{t}\left(d_tu + \frac{t}{x}.d_xu\right) = Pt^2;$$

$$\therefore\ \varDelta_t D_t u-(1\div t)D_t u = Pt^2,$$

$$\because\ D_t u = \left(d_t + \frac{t}{x}\cdot d_x\right)u.$$

Divide by t and integrate with Σ_t,

$$\therefore\ D_t u = t\Sigma_t(Pt)+tf(\xi');$$

Integrate again with S_t,

$$\therefore\ u = S_t t\Sigma_t(Pt)+S_t f(\xi')+F(\xi).$$

But $\quad S_t tf(\xi') = S_t tf(\xi+2t^2) = \varphi(\xi+2t^2) = \varphi(x^2+t^2);$

$$\therefore\ u = S_t t\Sigma_t Pt+F(x^2-t^2)+\varphi(x^2+t^2).$$

EXAMPLE 2. The integral of the equation

$$\frac{d^2u}{dt^2}-a^2\frac{d^2u}{dx^2}+b\left(\frac{du}{dt}+a\frac{du}{dx}\right) = 0,$$

is $u = F(x-at)+\varepsilon^{-bt}f(x+at)$. Show, by a simple method how this integral is obtained.

Solution.—Put the given equation into the following symbolic form, viz.;

$$d_t^2u-a^2d_x^2u+b(d_tu+ad_xu) = 0.$$

Then, since

$$d_t^2-a^2d_x^2 = (d_t+ad_x)(d_t-ad_x) = (d_t-ad_x)(d_t+ad_x),$$

we *may* use two sets of independent variables, corresponding to the two equations,

$$D_tu = d_tu + ad_xu,$$
$$\varDelta_t u = d_tu - ad_xu.$$

In these equations we *may* assume $D_tx = a$, and $\varDelta_tx = -a$, and since the symbols D_t, $\varDelta_t$, are respectively equivalent to the compound symbols d_t+ad_x, d_t-ad_x, the substitution of them in the proposed equation gives

$$D_t\varDelta_tu + bD_tu = 0.$$

Hence the following system of simultaneous equations is equivalent to the proposed equation.

$$\left.\begin{aligned} D_tx &= a \\ \varDelta_tx &= -a \\ D_t\varDelta_tu + b.D_tu &= 0. \end{aligned}\right\} \qquad (1)$$

The first and second of equations (1) give

$$x = at + \xi, \text{ suppose,}$$
$$x = -at + \xi', \text{ “}$$

while the third, being integrated with the symbol S_t, gives

$$\varDelta_tu + bu = F(\xi) = F(x-at). \qquad (2)$$

Now the symbols D_t and $\varDelta_t$ are commutative, because their equations d_t+ad_x, d_t-ad_x are so; consequently eq. (1) may be written in the form

$$\varDelta_tD_tu + b.D_tu = 0.$$

This equation being integrated with Σ_t gives

$$D_t u = \varepsilon^{-bt} f(\xi') = \varepsilon^{-bt} f(x+at),$$

which added to equation (2) and the sum equated to u, gives

$$u = F(x-at) + \varepsilon^{-bt} f(x+at),$$

for the required integral.

Proof.—That u may be written for $D_t u + \varDelta_t u + bu$ is manifest from the following operations, viz.:—As the proposed equation is represented by

$$D_t \varDelta_t u + b D_t u = 0, \qquad (\alpha)$$

let it be differentiated separately with the symbols D_t and $\varDelta_t$, and also multiplied by the constant b, thus;

$$\begin{aligned} D_t\varDelta_t . D_t u + bD_t . D_t u &= 0, \\ D_t\varDelta_t . \varDelta_t u + bD_t . \varDelta_t u &= 0, \\ D_t\varDelta_t . bu + bD_t . bu &= 0. \end{aligned}$$

Hence, since D_t and $\varDelta_t$ are commutative,

$$D_t\varDelta_t(D_t u+\varDelta_t u+bu)+bD_t(D_t u+\varDelta_t u+bu) = 0.$$

This equation is of the same form as equation (α), and we learn from it that $D_t u+\varDelta_t u+bu$ will satisfy the proposed equation as well as u will satisfy it. If therefore $D_t u+\varDelta_t u+bu$ contain the requisite number of arbit'y functions, it is a complete integral of the proposed equation, and, being so, we may write u for it.

EXAMPLE 3. Given the equations

$$d_t^2\psi+(a+b)d_t d_x\psi+abd_x^2\psi = tx, \qquad (\alpha)$$

$$t^2 d_t^2\psi + 2txd_t d_x\psi + x^2 d_x^2\psi = t^a x^b; \qquad (\beta)$$

to find their symbolic integrals.

Solution.—The late Prof. Peirce, in his admirable Integral Calculus, p. 275, proposed these equations as exercises, and solved the first for $P = tx$ by one of his own *unique* methods of solution. A solution, by somewhat different methods, is herewith offered in the hope that it will not be uninteresting to the readers of the ANALYST.

Of the first equation (α) it is well known that it consists of two independent parts; one of them, independent of $P=tx$, is called the absolute part of the integral; the other is dependent on P. These two parts, being independent of each other, may be found separately, and their sum will constitute the complete integral. The absolute part corresponds to the general supposition, $P = 0$.

Under the condition $P = 0$, equation (α) may take the form

$$(d_t + ad_x)(d_t + bd_x)\psi = 0,$$

in which we are at liberty to assume $D_t = d_t+ad_x$, $\varDelta_t = d_t + bd_x$, and $D_t x = a$, $\varDelta_t x = b$. The two last being integrated and increased by an arbitrary constant for each, give

$$x = at + \xi,\ x = bt + \xi'.$$
$$\therefore \xi = x - at,\ \xi' = x - bt.$$

Hence equation (α) may be written

$$D_t \varDelta_t \psi = P + Q, \text{ suppose. } (Q = 0.)$$

Integrating this last equation with S_t and Σ_t, we find

$$\begin{aligned}\psi &= S_t \Sigma_t P + S_t Q + \Sigma_t Q \\ &= S_t \Sigma_t P + F(\xi) + f(\xi') \\ &= S_t \Sigma_t P + F(x-at) + f(x-bt).\end{aligned}$$

Let the function P be expounded by tx; then as $D_t x = a$, and $\varDelta_t x = b$, we find

$$D_t(tx) = x + at,$$
$$\varDelta_t(tx) = x + bt.$$

If therefore we may put

$$S_t \varDelta_t P = \varDelta_t S_t P = \frac{(tx)^3}{2.3(x+at)(x+bt)},$$

the integral of equation (α) may be written

$$\psi = \frac{(tx)^3}{2.3(x+at)(x+bt)} + F(x-at) + f(x-bt).$$

Next, dividing equation (β) by t^2 we have

$$[d_t + (x \div t) . d_x]^2 = t^{a-2} x^b,$$
$$D_t = d_t + (x \div t) . d_x.$$

Hence, if we put $D_t^2 \psi = t^{a-2} x^b$, and $\xi = (x \div t)$, we shall have

$$D_t^2 \psi = \xi^b . t^{a+b-2};$$

which being integrated *twice* with S_t, and each integration increased by an arbitrary function of ξ, gives

$$\begin{aligned}\psi &= \frac{t^{a+b}}{(a+b)(a+b-1)} . \xi^b + t\,F(\xi) + f(\xi) \\ &= \frac{t^a x^b}{(a+b)(a+b-1)} + F\left(\frac{x}{t}\right) + f\left(\frac{x}{t}\right).\end{aligned}$$

This result holds good for any function of $x \div t$ written in the place of a and b.

EXAMPLE 4. It is required to integrate the equation

$$\left(\frac{d\psi}{dx}\right)^2 . \frac{d^2\psi}{dt^2} - 2\frac{d\psi}{dx}\frac{d\psi}{dt} . \frac{d^2\psi}{dt\,dx} + \left(\frac{d\psi}{dt}\right)^2 . \frac{d^2\psi}{dx^2} = P\left(\frac{d\psi}{dx}\right)^3,$$

(1) when $P = \frac{d\psi}{dt} \div \frac{d\psi}{dx}$, and (2) when $P = 0$.

Solution.—This equation includes examples (4) and (5) of Section 228, Vol. II, of Peirce's "Curves Functions and Forces", under a different

notation; and to bring it within the domain of more recent discussions on partial differential equations of the second order, let us denote

$$\frac{d\psi}{dt} \text{ by } p,\ \frac{d\psi}{dx} \text{ by } q,\ \frac{d^2\psi}{dt^2} \text{ by } d_t^2\psi, \text{ and } \frac{d^2\psi}{dx^2} \text{ by } d_x^2\psi.$$

Then, the given equation, expressed symbolically, gives

$$q^2 d_t^2\psi - 2pq d_t d_x\psi + p^2 d_x^2\psi = pq^2.$$

Divide this equation by q^2, then

$$d_t^2\psi - \frac{2p}{q} d_t d_x\psi + \frac{p^2}{q^2} d_x^2\psi = p. \tag{1}$$

If now we bear in mind that x is a function of t, and designate the total differential coefficient of p and q by $D_t p$ and $D_t q$, when p and q are differentiated with reference to the variables x and t, we shall find

$$D_t p = d_t p + d_x p . D_t x = d_t p + D_t x . d_x p = d_t^2\psi + D_t x . d_t d_x \psi, \tag{α}$$

$$D_t q = d_t q + d_x q . D_t x = d_t q + D_t x . d_x q = d_t d_x\psi + D_t x . d_x^2\psi. \tag{β}$$

Multiply equation (β) by $D_t x$ and add the product to equation (α); then

$$D_t p + D_t x . D_t q = d_t^2\psi + 2D_t x . d_t d_x\psi + (D_t x)^2 . d_x^2\psi. \tag{γ}$$

Comparing equation (1) with equation (γ) we find that

$$D_t x = -p \div q, \text{ and } D_t p + D_t x . D_t q = p.$$

By transposition and substitution, these equations give

$$p + D_t x . q = 0, \text{ and } D_t p - (p \div q) . D_t q = p; \ \therefore\ D_t \psi = p + D_t x . q = 0,$$

$$\frac{qD_t p - pD_t q}{q^2} = \frac{p}{q} = -D_t x.$$

Hence, by integration $\psi = \xi$, an undetermined constant, and

$$p \div q = -\varphi'(t) + F(\xi);$$

whence $D_t x = -p \div q$, gives $-Dx = -\varphi'(t)dt + F(\xi)dt.$

By integration we get

$$-x = -\varphi(t) + tF(\xi) + f(\xi),$$

$$\therefore\ 0 = x - \varphi(t) + tF(\xi) + f(\xi) = x - \varphi(t) + tF(\psi) + f(\psi). \tag{2}$$

The condition (2), viz., $P = 0$, is readily fulfilled by making $D_t p + D_t x . D_t q = 0$, from which we obtain

$$\frac{qD_t p - pD_t q}{q^2} = 0.$$

By means of this equation, and the condition $D_t\psi = 0$, we readily find

$$-D_t x = F(\xi); \text{ whence } -x = tF(\xi) + f(\xi);$$

$$\therefore\ 0 = x + tF(\xi) + f(\xi) = x + tF(\psi) + f(\psi), \tag{3}$$

a result that might have been obtained by making $\varphi(t) = 0$ in eq'n (2).

Equation (3) is the typical eq'n of a certain class of Ruled Surfaces.